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FREE OSCILLATIONS OF AN ATMOSPHERE IN WHICH TEMPERATURE INCREASES LINEARLY WITH HEIGHT

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SUMMARY

It is shown that when the temperature in the atmosphere increases linearly with height, the speed of propagation of long waves does not approach a limit with increasing wave length, as in the case of an atmosphere in which the temperature at great heights is assumed to be constant or decreasing, but increases linearly with the period. The group velocity ultimately also increases linearly with the period and becomes equal to half the phase velocity. The region of maximum energy of the oscillation is shifted to increasingly higher elevations as the period is increased. Whereas, in an atmosphere where the temperature gradient at great heights is negative or zero, the tides are similar to those in a uniform ocean of equivalent depth H (about 8 km, depending on the assumed vertical temperature distribution), the superposition of an outer envelope with a positive temperature gradient introduces a radical change into the nature of the tide. Insofar as it is still legitimate to refer the tides in such an atmosphere to those in an ocean of equivalent depth H, it can be said that H becomes a function of the period, which increases indefinitely with the period. The bearing of these results on the resonance theory of atmospheric tides is discussed.

INTRODUCTION

The fact that the solar semidiurnal tide in the atmosphere is about 100 times larger than the equilibrium value led Kelvin to the conclusion that the atmosphere possesses a free period of tidal oscillation of 12 solar hours. The resonance period of the atmosphere must be within 4 minutes of half a solar day in order to account for the observed amplification and for the fact that the moon, whose tidal force is more than twice as large as that of the sun, excites a barometric oscillation having an amplitude of only one-sixteenth that of the solar wave. If the vertical temperature distribution in the atmosphere, assumed horizontally stratified, is known, it is possible to compute its free period of tidal oscillation following a method due to the work of G. I. Taylor (reference 1). According to this method one first determines the speed of propagation of

long waves V in a flat atmosphere having the same vertical temperature distribution. The resonance period or periods of tidal oscillations of the atmosphere are then identical with those of an ocean of depth H enveloping the earth and are determined from the relation

$$V = (gH)^{1/2} \tag{1}$$

The dependence of the free period of the ocean on its depth H is known from classical tidal theory. For the solar semidiurnal oscillation $H = 7.87$ kilometers ± 3 percent, and therefore V must be 0.278 kilometer per second or $0.817c_0$, where c_0 denotes the velocity of sound in the air next to the ground, that is, 340 meters per second for an assumed surface temperature of 288° K. On assuming a factor $\exp i(\sigma t - kx)$ the divergence χ of the wave motion is in the limit of long waves found to satisfy the equation

$$c^2 \frac{d^2\chi}{dz^2} + \left(\frac{dc^2}{dz} - \gamma g\right)\frac{d\chi}{dz} + \frac{\chi}{H}\left[\frac{dc^2}{dz} + g(\gamma - 1)\right] = 0 \tag{2}$$

$$c^2 = \gamma RT$$

$$H = \sigma^2/gk^2$$

$$V = \sigma/k$$

subject to the boundary conditions that at the ground the vertical component of the velocity w vanishes:

$$w = \left(c^2\chi/g\right) + H\left[-\gamma\chi + \left(c^2/g\right)(d\chi/dz)\right] = 0 \tag{3}$$

and that the energy of wave motion per unit column of the atmosphere is not infinite. If $\rho(z)$ denotes the density of the air as a function of height z then of the two independent solutions of equation (2) the one is accepted which gives a finite value for the energy integral U

$$U = (1/2)\int_0^\infty \rho(z)\left(u^2 + v^2 + w^2\right)\,dz \tag{4}$$

where u, v, and w denote the components of velocity due to the wave motion.

When the temperature at great heights is assumed to be sufficiently low, or decreasing, it is actually found (reference 2) that one of the solutions of equation (2) gives a finite value for U, while in the other the distribution of $\left(u^2 + v^2 + w^2\right)$ with height is such that U becomes infinite as the integral is extended to infinity. When, however, the temperature at great heights is assumed to increase linearly without limit, it is found that for both solutions of equation (2) U becomes infinite, so that the question of the choice between the two solutions of equation (2) is left open. This is shown in a following section.

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SYMBOLS

A	constant in equation (63)
A, J, B, Y	coefficients in equations (8) and (9)
b	coefficient in equation (52)
$C = \sin \pi\delta/\sin \pi(\delta + n)$	
c	velocity of sound at any level
c_0	velocity of sound in air next to ground
$D = (2/\gamma) \times \left[(1 + n)\omega\right]^{1/2}$	
$E = (8\pi/g)(1 + n)c_0\left[(1 + n)\omega\right]^{-1/2}$	
F	confluent hypergeometric function
F_1, F_2, G	defined by equations (33)
F_3, F_4, N	defined by equations (34a)
$F(\theta)$	Airy function (reference 3)
g	acceleration of gravity
H	depth in a uniform ocean
I	Bessel function with imaginary argument

J Bessel function with real argument

$K = \Gamma(1 + \delta) \sin \pi(\delta + n)/2^{n+1} \sin \pi n$

k wave number

M coefficient in equation (47)

$m = \left[\gamma + (\gamma - 1)(1 + n)\right] / \gamma H$

$n = (g/R\beta) - 1$

P period of wave

P_B Brunt's period (reference 4)

$Q(u)$ function defined by equation (54)

R gas constant

T temperature

t time; in appendix B, the plane t

U energy integral

u, v, w components of velocity due to wave motion

V speed of propagation of long waves; phase velocity

$W(x)$ Whittaker's function (reference 5)

$W_{k,m}(z)$, $M_{k,m}(z)$ defined by equations (19) and (20)

x, z distances in x- and z-directions

x roots of equation (35)

x_0 value of x at ground

$y = 2\sqrt{\delta x} = 2\sqrt{mz}$

z_0 level above which temperature increases linearly with height

β constant in equation (5)

Γ function in equation (19)

γ ratio of specific heats

$\delta = -1 - (n/2) + \left[\tau(1 + n)/2\gamma\right] + (\omega/2\gamma\tau)$

ϵ function in equation (46)

$\theta = (1/6)\delta^{-1/2} \left|4\delta - x\right|^{3/2}$

ρ density of air

ρ_0 undisturbed density

σ frequency of wave

$\tau = \sigma^2/gk = kH$

τ_1, τ_2 roots τ of equation (16)

$\varphi_1 \equiv 2\sqrt{mz} - (n\pi/2) - (3\pi/4)$

$\varphi_2 \equiv 2\sqrt{mz} + (n\pi/2) + (\pi/4)$

χ divergence of wave motion

$\omega = \gamma + (\gamma - 1)(1 + n)$

APPARENT FAILURE OF ENERGY-INTEGRAL CRITERION FOR AN ATMOSPHERE IN WHICH TEMPERATURE INCREASES LINEARLY WITH HEIGHT

Let the temperature T increase linearly with height above the level z_0,

$$\left.\begin{aligned} T &= \beta z,\ z > z_0 \\ c^2 &= \gamma RT = \gamma gz/(1 + n) \\ n &= (g/R\beta) - 1 \end{aligned}\right\} \tag{5}$$

then one easily finds that the density decreases with height according to the power law

$$\rho/\rho_0 = \left(z/z_0\right)^{-n-2} \tag{6}$$

Equation (2) then takes on the form

$$\left(d^2\chi/dz^2\right) - (n/z)(d\chi/dz) + (m/z)\chi = 0 \tag{7}$$

$$m \equiv \left[\gamma + (\gamma - 1)(1 + n)\right] \Big/ \gamma H$$

whose solution is

$$\chi = Az^{(1+n)/2}J_{1+n}\left(2\sqrt{mz}\right) + Bz^{(1+n)/2}Y_{1+n}\left(2\sqrt{mz}\right) \tag{8}$$

when n is an integer, or

$$\chi = Az^{(1+n)/2}J_{1+n}\left(2\sqrt{mz}\right) + Bz^{(1+n)/2}J_{-1-n}\left(2\sqrt{mz}\right) \tag{9}$$

when n is not an integer.

In evaluating the energy integral U in equation (4) for the purpose of making the choice between the solutions in equation (8) or in equation (9), one has, for a wave progressing in the x-direction,

$$\left.\begin{aligned} u &= (i/gkH)\left(c^2\chi - gw\right) \\ v &= 0 \end{aligned}\right\} \tag{10}$$

Since $c^2\chi \approx z\chi$ while, for large values of z, $d\chi/dz \approx \chi$ it follows that at great heights both u and w vary like $z\chi$. Now for large values of z

$$J_{1+n}\left(2\sqrt{mz}\right) \approx \left(\pi^2 mz\right)^{-1/4} \cos \varphi_1 \tag{11}$$

$$\varphi_1 \equiv \left[2\sqrt{mz} - (n\pi/2) - (3\pi/4)\right]$$

$$J_{-1-n}\left(2\sqrt{mz}\right) \approx \left(\pi^2 mz\right)^{-1/4} \cos \varphi_2 \tag{12}$$

$$\varphi_2 \equiv \left[2\sqrt{mz} + (n\pi/2) + (\pi/4)\right]$$

It follows that $\left(u^2 + v^2 + w^2\right)$ in the integrand of equation (4) varies like $z^{1/2} \cos^2\varphi_1$ or $z^{1/2} \cos^2\varphi_2$ for the two solutions of equation (9), respectively. The kinetic energy per unit volume therefore increases ultimately with height like $z^{1/2}$ and the energy integral U does not converge for either solution. Not only does the energy-integral criterion not enable one to make the choice between the two solutions of equation (2) but also neither solution yields a finite value for U.

RESOLUTION OF DIFFICULTY WITH THE ENERGY INTEGRAL

The result obtained in the previous section that for an atmosphere in which the temperature increases linearly with height the energy integral of the wave motion diverges for both solutions of the wave equation is hard to reconcile with the following physical considerations. A medium such as the atmosphere, which is bounded by a perfectly reflecting surface at the ground but is exposed to space on the other side, would be expected to be facilitated in sustaining a free oscillation the more it is capable of diverting wave energy from great heights toward the ground. The more the upper atmosphere behaves like a reflector, the greater will be its trapping power for waves, and, with it, its ability to propagate waves to great ranges - a characteristic of free oscillations. Now it is known (references 6 to 8) that an atmosphere of constant temperature throughout can completely trap waves either in the whole spectrum or in a limited frequency range. This is also true for an atmosphere of the Taylor type in which the temperature is assumed to decrease at a constant rate in the troposphere and to remain constant in the stratosphere. Now when the temperature is constant, the rays emanating from a source situated in the atmosphere are rectilinear, and all but the horizontal rays are lost to space. In the Taylor type atmosphere the rays are even bent upward in the troposphere, eventually to become rectilinear in the stratosphere. In spite of these nonconservative properties, both types of atmosphere are capable of sustaining free oscillations in certain frequency ranges. In the case of an atmosphere with a linearly increasing temperature all the rays are bent downward, so that its trapping powers should be even greater. Yet, the divergence of the energy integral found in the previous section indicates a tendency to allow the waves to leak out to higher elevations.

The resolution of this paradox lies in the fact that equation (2) for the divergence X is not exact but is an approximation for the limiting case of long waves. This approximation breaks down when the temperature increases indefinitely linearly with height. The exact equation for X is equation (13)

$$\frac{d^2 X}{dz^2} + \left(\frac{1}{c^2}\frac{dc^2}{dz} - \frac{\gamma g}{c^2}\right)\frac{dX}{dz} + \left\{k^2\left(\frac{gH}{c^2} - 1\right) + \left[\frac{1}{H}\frac{1}{c^2}\frac{dc^2}{dz} + \frac{g(\gamma - 1)}{Hc^2}\right]\right\}X = 0 \qquad (13)$$

with

$$(1 - H^2k^2)w = (c^2\chi/g) + \left[H - \gamma\chi + (c^2/g)(d\chi/dz)\right] \tag{14}$$

replacing equation (3). Now in the limit of long waves, when the wave number k vanishes, the long-wave approximation is made by dropping the k^2-term in equation (13). This approximation is valid in an atmosphere of constant or decreasing temperature. In the case of an atmosphere in which the temperature (and with it c^2) increases linearly with height, the terms in brackets in equation (13) decrease like $1/z$ and eventually become smaller than the k^2-term. In fact at extremely large heights equation (13) reduces, for a given k, to

$$(d^2\chi/dz^2) - k^2\chi = 0 \tag{15}$$

indicating that the appropriate solution for χ ultimately varies like e^{-kz} and not according to the power law that would follow from equations (9), (11), and (12). When the complete equation (13) is used one finds that the solution having the factor e^{-kz} yields a finite value for the energy integral U in equation (4), whereas the other solution, with the factor e^{kz}, must be discarded because it gives a divergent value for U. With the complete wave equation (13) there is therefore no ambiguity as to the choice between the two independent solutions, nor is there any difficulty with the energy integral.

Using the notation of equations (5) and putting

$$\left.\begin{aligned} x &= 2kz \\ \tau &= \sigma^2/gk = kH \\ \delta &= -1 - (n/2) + \left[\tau(1 + n)/2\gamma\right] + (\omega/2\gamma\tau) \\ \omega &= \gamma + (\gamma - 1)(1 + n) \end{aligned}\right\} \tag{16}$$

equation (13) reduces to

$$x\left(d^2\chi/dx^2\right) - n(d\chi/dx) + \left[\delta + 1 + (n/2) - (x/4)\right]\chi = 0 \tag{17}$$

Choosing the solution of equation (17) which is bounded at infinity,

$$\chi = x^{n/2} W(x)_{\frac{(2\delta+2+n)}{2},\frac{(1+n)}{2}} \longrightarrow x^{1+\delta+n} e^{-x/2} \left\{1 - \left[\delta(\delta + 1 + n)/x\right] + \ldots\right\} \tag{18}$$

where $W(x)$ is Whittaker's function (reference 5, p. 337). The exponential behavior of χ for large values of x assures the convergence of integral (4) for U, whereas in the second solution of equation (17), which behaves like $x^{-1-\delta} e^{x/2}$ at infinity, the integral diverges.

LIMITING FORM OF SOLUTION FOR LONG WAVES

With the solution of the exact equation (13) given by equation (18) it becomes of interest to investigate the form it assumes in the limit of long wavelengths. In particular the relation of this limiting form of equation (18) to solution (9) of the approximate equation (2) is to be investigated, with a view of throwing light on the question of the choice of the proper linear combination of the two independent solutions appearing there. For this purpose the following relations (reference 5, pp. 337, 338, and 346) are used:

$$W_{k,m}(z) = \left[\Gamma(-2m)\middle/\Gamma\left(\frac{1}{2} - m - k\right)\right] M_{k,m}(z) + \left[\Gamma(2m)\middle/\Gamma\left(\frac{1}{2} + m - k\right)\right] M_{k,-m}(z) \tag{19}$$

$$\left.\begin{aligned} M_{k,m}(z) &= z^{\frac{1}{2}+m} e^{-z/2} F\left(\frac{1}{2} + m - k,\ 2m + 1,\ z\right) \\ M_{k,-m}(z) &= z^{\frac{1}{2}-m} e^{-z/2} F\left(\frac{1}{2} - m - k,\ 1 - 2m,\ z\right) \end{aligned}\right\} \tag{20}$$

where F denotes the confluent hypergeometric function. Thus, for the case when n is not an integer there is obtained

$$\begin{aligned} \chi(x) = {} & \left[\Gamma(-1 - n)/\Gamma(-\delta - n - 1)\right] x^{1+n} e^{-x/2} F(-\delta,\ 2 + n,\ x) + \\ & \left[\Gamma(1 + n)/\Gamma(-\delta)\right] e^{-x/2} F(-\delta - n - 1 - n,\ x) \end{aligned} \tag{21}$$

$$\chi(x) = \left\{\left[\Gamma(2 + \delta + n) \sin \pi(\delta + n)\right] \Big/ \left[\Gamma(2 + n) \sin \pi n\right]\right\} x^{1+n} e^{-x/2} F(-\delta, 2 + n, x) - \left[\Gamma(1 + n) \times \Gamma(1 + \delta) \sin (\pi\delta)/\pi\right] e^{-x/2} F(-\delta - 1 - n, -n, x) \tag{22}$$

where in the last transformation the following relation was used

$$\Gamma(x) \times \Gamma(1 - x) = \pi/\sin \pi x \tag{23}$$

Passing now to the limit of long waves,

$$\left.\begin{aligned} \tau &= kH \ll 1 \\ \delta &\rightarrow \omega/2\gamma\tau = m/2k \gg 1 \end{aligned}\right\} \tag{24}$$

With δ very large, the F-functions in equation (22) reduce to Bessel functions:

$$F(-\delta, 2 + n, x) \rightarrow \left[1 - \delta x/(2 + n) + (\delta x)^2/2(2 + n)(3 + n) + \cdot \cdot \cdot\right] = \Gamma(2 + n)(\delta x)^{-(n+1)/2} J_{n+1}\left(2\sqrt{\delta x}\right) \tag{25}$$

$$F(-\delta - 1 - n, -n, x) \rightarrow \Gamma(-n)(\delta x)^{(n+1)/2} J_{-n-1}\left(2\sqrt{\delta x}\right) \tag{26}$$

It is important to observe that these limiting forms of the F's are valid only when both $\delta \gg 1$ and $x \ll \delta$, for otherwise the series for F depends on the higher powers of x, when it is no longer legitimate to replace the factor $\delta - k$ by δ in the coefficients. Substituting now equations (25) and (26) into equation (22) and performing some further reductions with the aid of equation (23), there is obtained

$$\chi = K \exp\left[-(y^2/8\delta)\right]\left[y^{n+1} J_{n+1}(y) + C y^{n+1} J_{-n-1}(y)\right] \tag{27}$$

$$\left.\begin{aligned} y &= 2\sqrt{\delta x} = 2\sqrt{mz} \\ K &= \Gamma(1 + \delta) \sin \pi(\delta + n)/2^{n+1} \sin \pi n \end{aligned}\right\} \tag{28}$$

$$C = \sin \pi\delta/\sin \pi(\delta + n) \tag{29}$$

Except for the factor $\exp\left[-(y^2/8\delta)\right]$, which is slowly variable in the limit of large values of δ, equation (27) is identical with equation (9). The ratio of constants B/A in the latter is now, however, no longer arbitrary but is a definite function C of δ, as given in equation (29).

The goal of determining the appropriate solution of the wave equation for an atmosphere with a linear increase of temperature has thus been achieved, but on closer study the physical implications of the results are rather disturbing. In order to ascertain the physical significance of the parameter δ the phase velocity and the period are determined. The phase velocity V is given by

$$V/c_0 = \sigma/kc_0 = (2/\gamma) \times \left[(1 + n)\omega\right]^{1/2}(1/y) \equiv D/y \tag{30}$$

while the period of the wave P, measured in seconds, is given by

$$P = 2\pi/\sigma = (8\pi/g)(1 + n)c_0\left[(1 + n)\omega\right]^{-1/2}(\delta/y) \equiv E\delta/y \tag{31}$$

For $n = 4.5$, then $D = 6.36$ and $E = 1080$, while for $n = 8.5$, which is appropriate for the positive lapse rate prevailing in the NACA model atmosphere between 83 and 120 kilometers (reference 9), $D = 10.0$ and $E = 1180$. For V/c_0 of the order of unity, the period P changes by about 3 minutes as δ changes by unity. But, since $C(\delta)$ in equation (29) is a periodic function of δ of period unity and takes on all values between $-\infty$ and ∞ as δ changes by 1, it follows that in the limit of long periods X does not approach a fixed form but changes its character radically as the period is changed by only a few minutes. This increased sensitivity of X to the period for an atmosphere with positive lapse rate is to be contrasted with the decreasing sensitivity to changes in period for atmospheres with zero or negative lapse rates.

FREE OSCILLATIONS OF AN ATMOSPHERE IN WHICH TEMPERATURE INCREASES LINEARLY WITH HEIGHT

In order to gain more insight into the acoustic propagating properties of an atmosphere with a constant positive temperature gradient, the free oscillations of such an atmosphere when it extends from the ground up will be investigated. Therefore the variation of the phase velocity V with the period P for the first and for the higher normal modes will be studied. Such a variation is shown, for example, by curve I in figure 2, for the first mode of an atmosphere with constant negative temperature gradient (references 8 and 10). For a Taylor type atmosphere having the same negative temperature gradient in the troposphere, the corresponding curve starts with a value of 0.86 at $P = 2$ minutes and reaches its limiting value of about 0.92 at $P = 4$ minutes (references 8 and 10).

The divergence χ from equation (22) can be written in the form

$$\chi = x^{1+n}e^{-x/2}F_1 - Ge^{-x/2}F_2 \tag{32}$$

with

$$\left.\begin{aligned} G &= \left[\Gamma(1+n)\Gamma(2+n)\Gamma(1+\delta)\ \sin \pi n \times \sin \pi\delta\right] \Big/ \left[\pi\Gamma(2+\delta+n) \times \sin \pi(\delta+n)\right] \\ F_1 &= F(-\delta,\ 2+n,\ x) \\ F_2 &= F(-\delta-1-n,\ -n,\ x) \end{aligned}\right\} \tag{33}$$

The possible free oscillations of the atmosphere are determined from the condition of the vanishing of the vertical component of velocity w at the ground, which, according to equation (14), is given by

$$(1+n)(1-\tau^2)(w/\gamma H\chi) = -1 - n + (x/2\tau) + (x/\chi)(d\chi/dx) = 0 \tag{34}$$

With

$$\left.\begin{aligned} F_3 &= F(1-\delta,\ 3+n,\ x) \\ F_4 &= F(-\delta-n,\ 1-n,\ x) \\ D &= F_1 - Gx^{-1-n}F_2 \\ N &= \left[(1+n)/x\right]F_1 - \left[\delta/(2+n)\right]F_3 - G\left[(\delta+1+n)/n\right]x^{-1-n}F_4 \end{aligned}\right\} \tag{34a}$$

equation (34) leads to the condition

$$\left.\begin{aligned}(1 + n)\left(1 - \tau^2\right)(w/\gamma HX) = -(1 + n) + (x/2)\left[(1/\tau) - 1\right] + xN/D = 0 \\ x = x_0\end{aligned}\right\} \quad (35)$$

where x_0 denotes the value of x at the ground.

The problem of determining the dependence of the phase velocity V on the period P involves the determination of the dependence of the roots x of equation (35) on the parameter δ. For a given δ one first determines the two roots τ of equation (16):

$$\tau_1 = \left(\gamma\left[\delta + 1 + (n/2)\right] + \left\{\gamma^2\left[\delta + 1 + (n/2)\right]^2 - (1 + n)\omega\right\}^{1/2}\right)\Big/(1 + n)$$

$$\tau_2 = \left(\gamma\left[\delta + 1 + (n/2)\right] - \left\{\gamma^2\left[\delta + 1 + (n/2)\right]^2 - (1 + n)\omega\right\}^{1/2}\right)\Big/(1 + n) \quad (37)$$

For each τ one finds the roots x of equation (35) and determines V and P from

$$V/c_0 = \left[2(1 + n)\tau/\gamma x\right]^{1/2} \quad (38)$$

$$P = (2\pi c_0/g)\left[2(1 + n)/\gamma\tau x\right]^{1/2} \quad (39)$$

The free oscillations of an atmosphere have been determined with $n = 4.5$, corresponding to a positive temperature gradient of about 6.4° per kilometer. Figure 1 shows the variation of the roots x with δ for the two branches τ_1 and τ_2. It is shown in appendix A that for the first mode in the τ_2-branch, which is shown by curve A in figure 1, the lower end of the curve is given by

$$x = 1.88(1 - \delta)^{1/6.5} \quad (40)$$

for sufficiently small values of x. With $\tau_2 = 4/11$, it follows from equations (38) and (39) that along this end of curve A, $P \rightarrow \infty$, $V/c_0 \rightarrow \infty$, and $V/c_0 \rightarrow (1/10)P$, when P is measured in minutes. Hence in an atmosphere with constant positive temperature gradient, the phase velocity does not approach a limit with increasing period but increases

linearly with the period. The same result applies to the higher modes. Figure 2 shows the function $V(P)$ which results from the $x(\delta)$-curves in figure 1. It is shown in appendix B that in the τ_2-branch no waves can be propagated for periods less than 4.5 minutes, which is Brunt's period (reference 4) determined from

$$c^2\sigma^2 = g^2(\gamma - 1) + g\left(dc^2/dz\right) \tag{41}$$

In all the modes V/c_0 grows indefinitely with the period.

It is of interest to see the type of dispersion one gets by using the "long wave" solution (27). Dropping the exponential term,

$$\chi = y^{n+1}J_{n+1}(y) + Cy^{n+1}J_{-n-1}(y) \tag{42}$$

$$d\chi/dy = y^{n+1}J_n(y) - Cy^{n+1}J_{-n}(y) \tag{43}$$

while the secular equation (35) reduces to

$$-(1 + n) + \left(\gamma y^2/4\omega\right) + (y/2)\left[J_n(y) - CJ_{-n}(y)\right]\Big/\left[J_{n+1}(y) + CJ_{-n-1}(y)\right] = 0 \tag{44}$$

In this case one can solve directly for C, since all the other terms do not depend explicitly on δ:

$$C = \left\{J_n(y) - \left[(1 + n)(2/y) - (\gamma y/2\omega)\right]J_{n+1}(y)\right\}\Big/\left\{J_{-n}(y) + \left[(1 + n)(2/y) - (\gamma y/2\omega)\right]J_{-n-1}(y)\right\} \tag{45}$$

With $C(\delta)$ given in equation (29), equation (45) allows one to obtain δ as a function of y, from which one determines the function $V(P)$ by using equations (30) and (31). The results are shown in figure 3.

It is seen that the general shape of the $V(P)$-curves is similar to that of the exact curves given in figure 2. The deviations are due to the fact that, in the upper right-hand corner, δ is of the order of unity, for which the long-wave approximation cannot apply. The curves in figure 3 become also inaccurate as the cut-off period is approached,

because there y is of the order of δ, and the Bessel function approximation in relations (25) and (26) breaks down. The analysis for this limiting case is given in appendix B.

APPLICATION TO THEORY OF ATMOSPHERIC TIDES

This investigation has shown that in an atmosphere where the temperature increases indefinitely with height at a constant rate, the acoustic propagation properties are radically different from those that have hitherto been encountered in the theory of atmospheric tides. The principal new feature is that the phase velocity V no longer reaches a limit for waves of long period. For atmospheres where V reaches a limit, the theory of atmospheric tides is simplified by an application of Taylor's theorem (reference 1), according to which the tides in the atmosphere are equivalent to those in an ocean of uniform depth $H = V^2/g$. In our case, V grows indefinitely with the period, and, insofar as it is still legitimate to refer to an equivalent ocean of depth H, one may say that H depends on the period, ultimately increasing as the square of the period. What happens is that, as the period is increased, the distribution of energy of wave motion with height no longer approaches a fixed form, but the energy continues to spread to higher elevations as the period is increased. A discussion will not be given here of the nature of the tides in an atmosphere with constant positive temperature gradient extending from the ground up. In such an atmosphere the tides are probably different from those in an ocean of fixed depth. An investigation of this problem would be in order if the evidence would continue to accumulate to the effect that the temperature in the ionosphere increases continuously with height.

Pending such an investigation, suffice it to remark that there is no difficulty in treating the case where a layer with positive temperature gradient is topped by an outer envelope of constant or decreasing temperature. In that case the proper solution for the layer is given by equation (9), with B/A arbitrary. It would also follow that in the absence of proof as to the temperature of the outer envelope of the atmosphere one should not in theoretical studies allow an assumed positive gradient in an interior layer to extend through the outer envelope. The consequences of such an extrapolation of the temperature curve, if rigorously determined, are, in the light of the results of this investigation, likely to affect seriously the nature of the atmospheric tides. Any preliminary results obtained for an atmosphere with a positive temperature gradient extending through the envelope should therefore be checked by comparing them with those for a modified atmosphere in which, say, the temperature in the outer envelope is assumed to be constant or decreasing with height. It would also be helpful to compute in each case the distribution with height of the energy density of the wave

motion $(\rho/2)\left(u^2 + v^2 + w^2\right)$ in order to assure that any segment of the temperature curve which is as yet not supported by observations does not effect a marked redistribution in the wave energy.

Institute for Advanced Study
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APPENDIX A

DEPENDENCE OF PHASE VELOCITY V ON PERIOD P WHEN PERIOD IS INCREASED INDEFINITELY

Only the τ_2-branch of the first mode will be treated here and in the particular case when $n = 4.5$. The higher-order modes, as well as those of the τ_1-branch, can be dealt with similarly. The problem is to determine the limiting form of the root $x(\delta)$ of equation (35) as $\delta \longrightarrow 1 - \epsilon$. Then

$$\left.\begin{aligned} &\tau_2 = 4/11 \\ &G \longrightarrow -\epsilon\Gamma(5.5)/6.5 \end{aligned}\right\} \tag{46}$$

Also for a vanishing x all the F's reduce to unity. Now assume that

$$G \longrightarrow Mx^{6.5} \tag{47}$$

as $x \longrightarrow 0$, and seek to determine the coefficient M from equation (35). On retaining only leading terms, equation (35) becomes

$$-5.5 + 0.875x + \left[5.5 - (x/6.5)\right](1 + Mx) = 0 \tag{48}$$

On annulling the coefficient of x there is obtained

$$M = -0.721 = -(1 - \delta)\left[\Gamma(5.5)/6.5\right]x^{-6.5} \tag{49}$$

$$x = 1.88(1 - \delta)^{1/6.5} \tag{50}$$

APPENDIX B

DETERMINATION OF CUT-OFF PERIOD

If equation (45) is evaluated for large values of y by using the asymptotic forms of the Bessel functions it is found that all modes coalesce at a limiting period (5.7 min for $n = 4.5$), for which the phase velocity vanishes. This result can, however, not be relied upon since when y becomes of the order of δ the Bessel function approximations in relations (25) and (26) are no longer valid. In order to determine the cut-off period the asymptotic form of W in equation (18) has to be obtained when both δ and x become large and are of the same order of magnitude. For this purpose Whittaker's (reference 5, p. 339) countour-integral representation for W is used, which can be put in the form

$$\begin{aligned} \chi &= x^{n/2}W(x) \\ &= (1/2\pi i)e^{i\pi\delta}\Gamma(1 + \delta)e^{-x/2}x^{1+n}\oint t^{-\delta-1}(1 + t)^{\delta+1+n}e^{-xt}\,dt \end{aligned} \tag{51}$$

the path starting at ∞ and returning to ∞ after encircling the origin in the positive direction, but excluding the point $t = -1$. Now put

$$\left.\begin{aligned} x &= 4\delta - b\delta \\ t &= u - (1/2) \end{aligned}\right\} \tag{52}$$

then near $t = -(1/2)$ there is the expansion

$$t^{-\delta-1}(1 + t)^{\delta+1+n}e^{-xt} = \left[\left(u + \frac{1}{2}\right)^{1+n}\Big/\left(u - \frac{1}{2}\right)\right]\exp\left[-i\pi\delta + (x/2) + Q(u)\right] \tag{53}$$

$$Q(u) = (\delta b)u + (16\delta/3)u^3 + (64\delta/5)u^5 + \dots \tag{54}$$

$$\chi = (1/2\pi i)\Gamma(1 + \delta)x^{1+n}\oint\left[\left(u + \frac{1}{2}\right)^{1+n}\Big/\left(u - \frac{1}{2}\right)\right]\exp Q(u)\,du \tag{55}$$

where the path in the u-plane is similar to the former path in the t-plane. For large values of δ and for values of b of the order of $\delta^{-2/3}$,

the principal contribution to the integral in equation (55) comes from the immediate vicinity of the origin. This can be made evident by putting

$$\left.\begin{aligned} (16\delta/3)u^3 &= v^3 \\ A &= (3/16)^{1/3}b\delta^{2/3} \end{aligned}\right\} \tag{56}$$

when the u^5-term in equation (54) becomes of the order of $\delta^{-2/3}$, while the expression in brackets in the integrand of equation (55) reduces to -2^{-n} to within terms of the order of $\delta^{-1/3}$. Hence,

$$\chi \approx -(1/2\pi i)2^{-n}x^{1+n}\Gamma(1 + \delta)(3/16\delta)^{1/3}\oint \exp\left(v^3 + Av\right) dv \tag{57}$$

Here the path starts at $\infty e^{i\pi/3}$ and terminates at $\infty e^{-i\pi/3}$ after encircling the origin in the positive direction. The integral can be evaluated in terms of the Airy function (reference 3, p. 189), yielding

$$\left.\begin{aligned} \chi &\approx 2^{-n}x^{1+n}\Gamma(1 + \delta)(288\delta)^{-1/3}F(\theta) \\ \theta &= (1/6)\delta^{-1/2}\left|4\delta - x\right|^{3/2} \end{aligned}\right\} \tag{58}$$

$$\left.\begin{aligned} F(\theta) &= \theta^{1/3}\left[J_{-1/3}(\theta) + J_{1/3}(\theta)\right] \\ x &< 4\delta \end{aligned}\right\} \tag{59}$$

$$\left.\begin{aligned} F(\theta) &= \theta^{1/3}\left[I_{-1/3}(\theta) - I_{1/3}(\theta)\right] \\ x &> 4\delta \end{aligned}\right\} \tag{60}$$

Equation (60) shows that beyond 4δ, $\chi(x)$ is no longer oscillatory. The largest zeros of χ are determined from the zeros of $F(\theta)$ (reference 3, p. 751) in equation (59):

$$\left.\begin{aligned} \theta_1 &= 2.383 \\ \theta_2 &= 5.610 \\ \theta_3 &= 8.657 \end{aligned}\right\} \tag{61}$$

On substituting expression (59) into the secular equation (35) one finds that in the limit of large values of δ the roots of equation (35) coincide with the roots of χ. Hence the largest roots of equation (35) are given by

$$x_i = 4\delta - \left(36\delta\theta_i{}^2\right)^{1/3} \tag{62}$$

where the θ_i's are given in equation (61). From equation (62) it follows that near the cut-off period

$$\left.\begin{aligned} V/c_0 &\rightarrow (A/\delta)\left[1 + (1/8)(6\theta_i/\delta)^{2/3}\right] \\ P &= P_B\left[1 + (1/8)(6\theta_i/\delta)^{2/3}\right] \end{aligned}\right\} \tag{63}$$

where A is a constant and P_B denotes Brunt's period (reference 4).

REFERENCES

1. Taylor, G. I.: The Oscillations of the Atmosphere. Proc. Roy. Soc. (London), ser. A, vol. 156, Aug. 17, 1936, pp. 318-326.

2. Pekeris, C. L.: Atmospheric Oscillations. Proc. Roy. Soc. (London), ser. A, vol. 158, no. 895, Feb. 3, 1937, pp. 650-671.

3. Watson, G. N.: A Treatise on the Theory of Bessel Functions. Second ed., The Macmillan Co., 1944, p. 189.

4. Brunt, D.: The Period of Simple Vertical Oscillation in the Atmosphere. Quart. Jour. Roy. Meteorol. Soc., vol. 53, no. 221, Jan. 1927, pp. 30-32.

5. Whittaker, E. T., and Watson, G. N.: A Course of Modern Analysis. Fourth ed., Cambridge Univ. Press (London), 1927 (Reprinted 1940).

6. Lamb, Horace: Hydrodynamics. Sixth ed., Cambridge Univ. Press, 1932, p. 550.

7. Taylor, G. I.: Waves and Tides in Atmosphere. Proc. Roy. Soc. (London), ser. A, vol. 126, no. 803, March 3, 1930, pp. 169-183, 728.

8. Pekeris, C. L.: Propagation of Pulse in Atmosphere - II. Phys. Rev., vol. 73, no. 2, Jan. 15, 1948, pp. 145-154.

9. Warfield, Calvin N. (for the NACA Special Subcommittee on the Upper Atmosphere): Tentative Tables for the Properties of the Upper Atmosphere. NACA TN 1200, 1947.

10. Solberg, H.: Schwingungen und Wellenvewegungen in einer Atmosphäre mit nach oben abnehmender Temperature. Astrophysica Norvegica, vol. 2, no. 2, Aug. 1936, pp. 123-172.

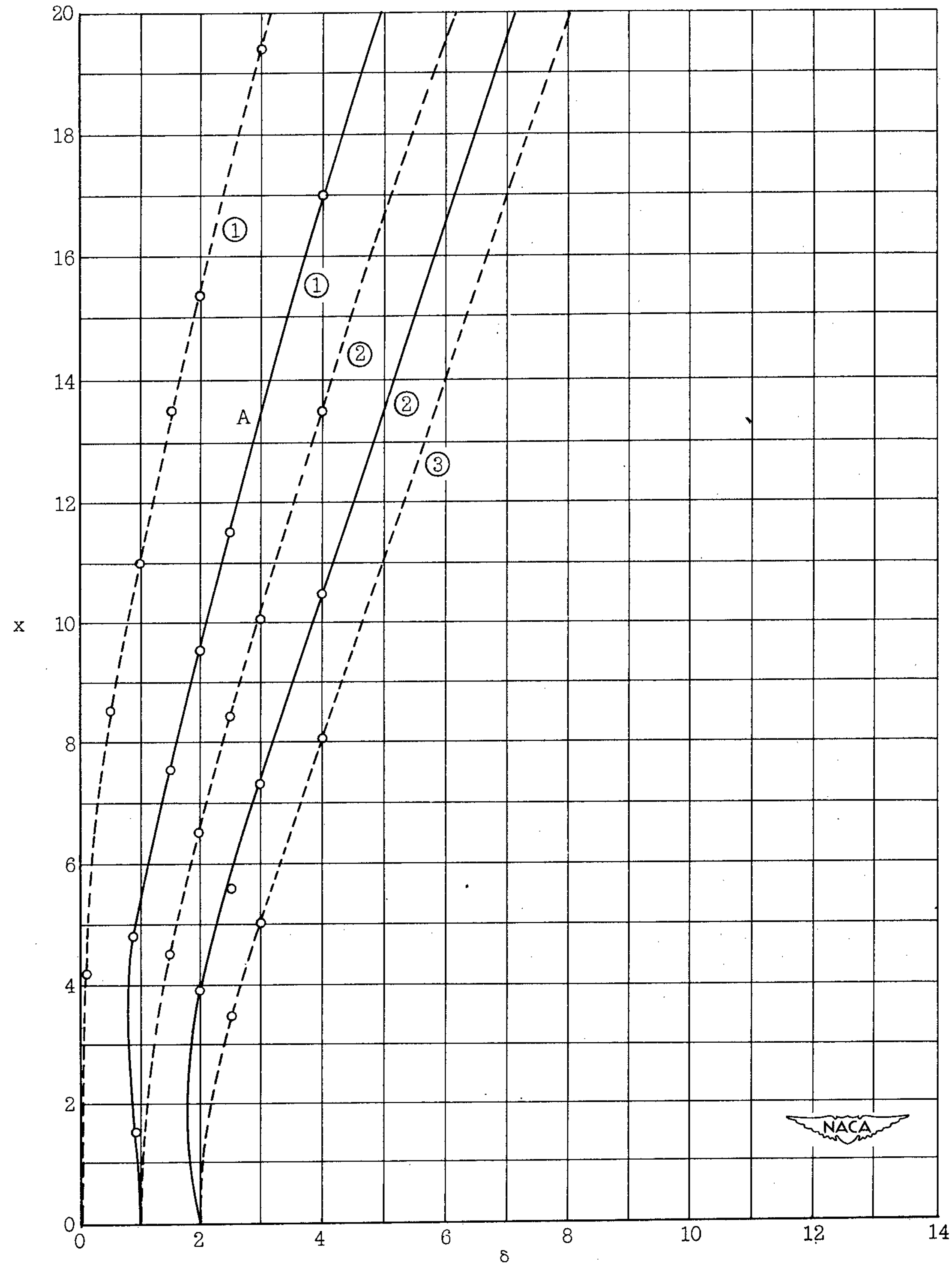

Figure 1.- Dependence of roots x of equation (35) on δ for an atmosphere with constant positive temperature gradient, $n = 4.5$. Solid curves for τ_2; dashed curves for τ_1. Encircled numbers refer to order of mode.

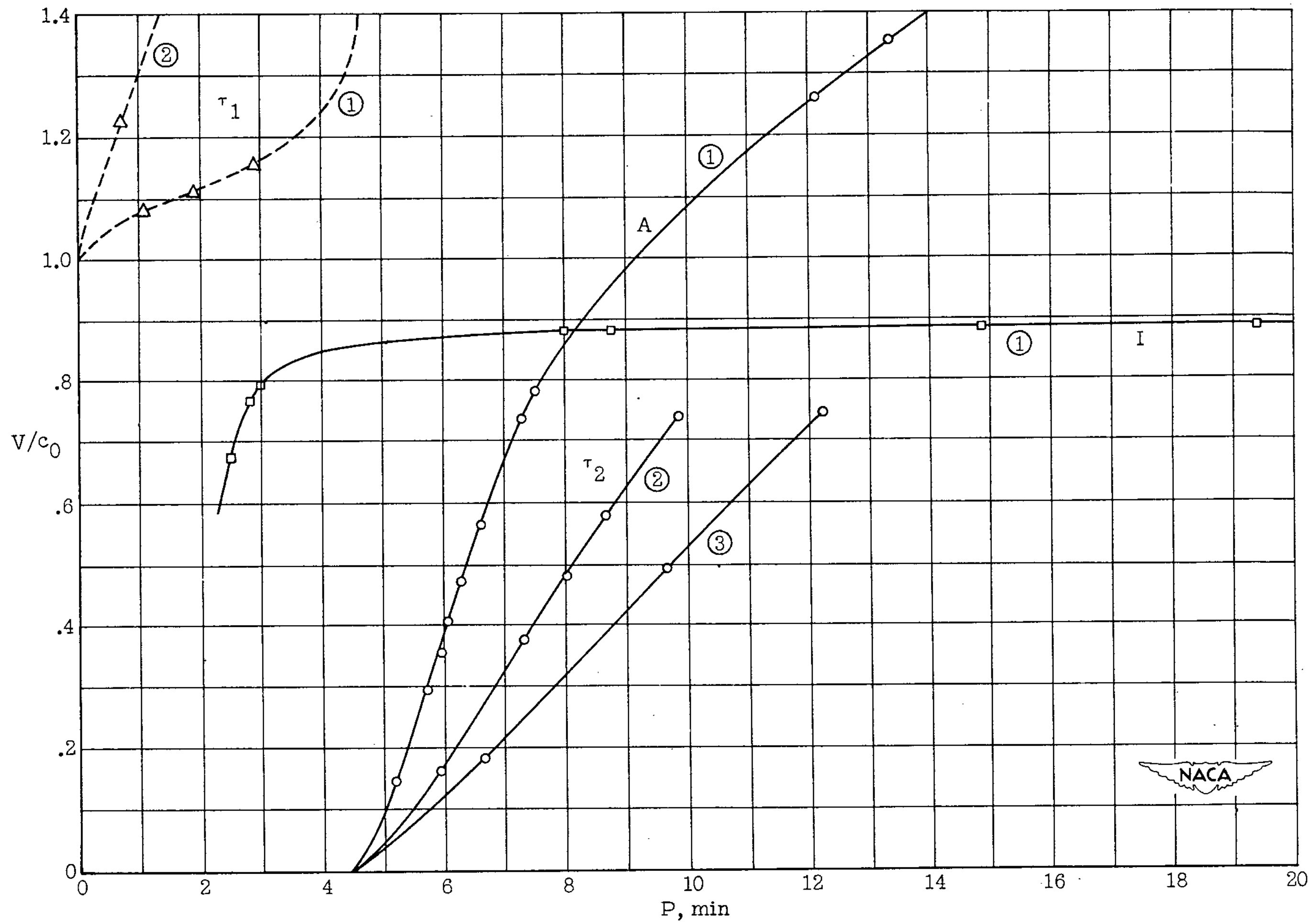

Figure 2.- Ratios of phase velocity V to velocity of sound near ground c_0 as a function of period P for an atmosphere with constant positive temperature gradient, n = 4.5. The τ_2-modes start at Brunt's period of 4.5 minutes. Curve I is for an atmosphere with constant negative temperature gradient of seven-elevenths of the adiabatic. Encircled numbers refer to order of mode.

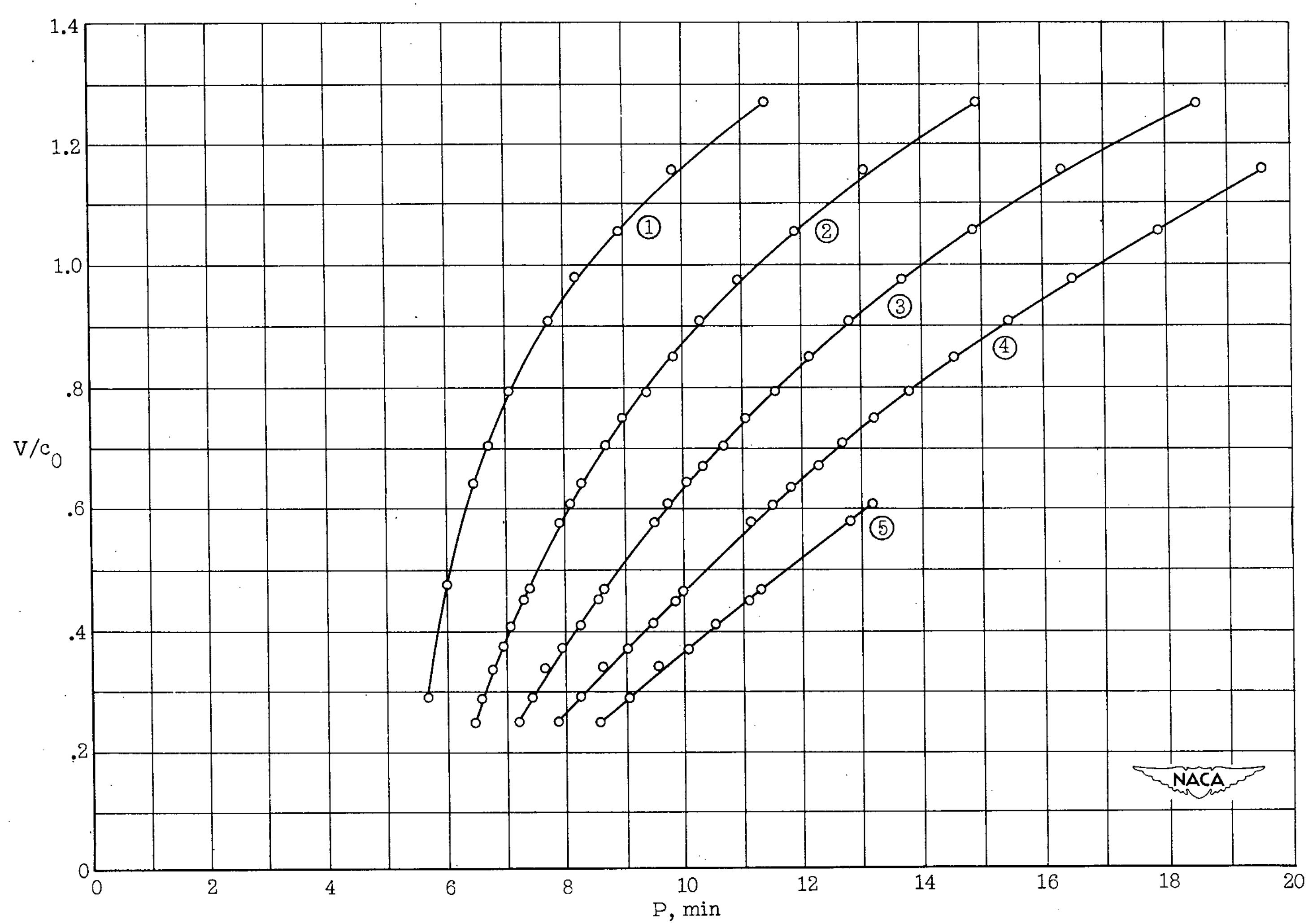

Figure 3.- Dependence of phase velocity V on period P obtained by using long-wave equations (45), (30), and (31) for an atmosphere with constant positive temperature gradient, n = 4.5. Encircled numbers refer to order of mode.